A Collocation Method for Solving Abel's Integral Equations of First and Second Kinds

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A numerical technique is developed for solving Abel's integral equations. The solutions of such equations may exhibit a singular behaviour in the neighbourhood of the initial point of the interval of integration. The proposed method is based on the shifted Legendre collocation technique. Illustrative examples are included to demonstrate the validity and applicability of the presented technique.

Key words: Volterra Integral Equation; Abel's Integral Equations; Shifted Legendre Polynomials; Collocation Method.

1. Introduction

In recent years, many different methods have been used to approximate the solution of Volterra integral equations with weakly singular kernels (see, for example, [1 – 3]). In the present paper, we consider the following Volterra integral equations of the first and second kinds, respectively:

$$\lambda \int_0^x \frac{y(t)}{(x-t)^{\alpha}} \mathrm{d}t = f(x), \quad 0 \le t \le x \le 1, \tag{1}$$

$$y(x) + \lambda \int_0^x \frac{y(t)}{(x-t)^{\alpha}} \mathrm{d}t = f(x), \quad 0 \le t \le x \le 1, \tag{2}$$

where $f(x)$ is in $L^2(R)$ on the interval $0 \le x \le 1$ and $0 < \alpha < 1$. Here λ, α and the function $f(x)$ are given, and $y(x)$ is the solution to be determined. For $0 < \alpha < 1$ the integral equations (1) and (2) are weakly singular and called Abel's integral equations of the first and second kinds, respectively. The special case $\alpha = 1/2$ often arises in physical problems. We assume that (1) and (2) have a unique solution.

In 1823, Abel, when generalizing "the tautochrone problem", derived (1). This equation is a particular case of a linear Volterra integral equation of the first kind. Abel's integral equations frequently appear in many physical and engineering problems, e. g., semiconductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reactions and population dynamics [4]. Many different authors presented numerical solutions for Abel's integral equation of first kind (see, for example, [5 – 7] and the references therein). In [6] the authors developed high-accuracy mechanical quadrature methods and, to avoid the ill-posedness of the problem, the first kind Abel integral equation was transformed to the second kind Volterra integral equation with a continuous kernel and a smooth right-hand side term expressed by weakly singular integrals. Also the author of [7] developed a numerical technique based on Legendre wavelet approximations for solving (1) and (2). The numerical treatment is more difficult for first kind than for second kind Abel integral equations, which have been widely studied [8 – 11].

In the present paper, we apply the shifted Legendre collocation method for solving Abel's integral equations. Our method consists of reducing Abel's integral equation to a set of linear algebraic equations by expanding the approximate solution as shifted Legendre polynomials with unknown coefficients. The properties of shifted Legendre polynomials are then utilized to evaluate the unknown coefficients.

The paper is organized as follows: In Section 2 we describe the basic formulation of Legendre and shifted Legendre polynomials required for our subsequent development. In Section 3 the application of the shifted Legendre collocation method to the solution of (1) and (2) is summarized. As a result a set of algebraic equations is formed and a solution of the considered problem is introduced. In Section 4 the proposed method is applied to numerical examples and the accuracy of our method using several examples is checked. Section 5 ends with a conclusion.

2. Shifted Legendre Polynomials

The well known Legendre polynomials are defined on the interval $z \in [-1,1]$ and can be determined with the help of the following recurrence formulae [12]:

$$L_0(z) = 1, \quad L_1(z) = z,$$
$$L_{i+1}(z) = \frac{2i+1}{i+1} z L_i(z) - \frac{i}{i+1} L_{i-1}(z),$$
$$i = 1,2,\dots.$$

In order to use these polynomials on the interval $x \in [0,1]$ we define the so-called shifted Legendre polynomials by introducing the change of variable $z = 2x-1$. Let the shifted Legendre polynomials $L_i(2x-1)$ be denoted $P_i(x)$. Then $P_i(x)$ can be obtained as follows:

$$P_0(x) = 1, \quad P_1(x) = 2x-1,$$
$$P_{i+1}(x) = \frac{(2i+1)(2x-1)}{(i+1)} P_i(x) - \frac{i}{i+1} P_{i-1}(x), \quad (3)$$
$$i = 1,2,\dots.$$

The analytical form of the shifted Legendre polynomial $P_i(x)$ of degree i is given by

$$P_i(x) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!} \frac{x^k}{(k!)^2}. \quad (4)$$

Note that $P_i(0) = (-1)^i$ and $P_i(1) = 1$. A function $u(x)$, on $[0,1]$, may be approximated in the form of a series with $n+1$ terms as

$$u(x) = \sum_{i=0}^{n} c_i P_i(x),$$

where the coefficients c_i $(i = 0,\dots,n)$ are constants.

3. Solution of the Singular Volterra Integral Equation

In this section we solve the singular Volterra integral (1) and (2) by using the shifted Legendre collocation method. First of all we approximate $y(x)$ as

$$y_n(x) = A_0 x^\alpha + \sum_{i=0}^{n} c_i P_i(x), \quad (5)$$

where A_0 and the coefficients c_i $(i = 0,\dots,n)$ are unknown. Substituting (5) into (1) we have

$$\lambda A_0 \int_0^x \frac{t^\alpha}{(x-t)^\alpha} \mathrm{d}t + \lambda \sum_{i=0}^{n} c_i \int_0^x \frac{P_i(t)}{(x-t)^\alpha} \mathrm{d}t = f(x). \quad (6)$$

Now, we know that

$$\int_0^x \frac{t^n}{(x-t)^\alpha} \mathrm{d}t = \frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma(n+2-\alpha)} x^{n+1-\alpha} \quad (7)$$

and

$$\int_0^x \frac{t^\alpha}{(x-t)^\alpha} \mathrm{d}t = \frac{\pi\alpha}{\sin(\pi\alpha)} x. \quad (8)$$

Employing (4) and (7) we obtain

$$\int_0^x \frac{P_i(t)}{(x-t)^\alpha} \mathrm{d}t = \sum_{k=0}^{i} a_{ik}^{(\alpha)} x^{k+1-\alpha}, \quad (9)$$

where

$$a_{ik}^{(\alpha)} = (-1)^{i+k} \frac{(i+k)!\Gamma(k+1)\Gamma(1-\alpha)}{(k!)^2 (i-k)!\Gamma(k+2-\alpha)} = (-1)^{i+k} \frac{(i+k)!\Gamma(1-\alpha)}{k!(i-k)!\Gamma(k+2-\alpha)}. \quad (10)$$

By using (8) and (9), (6) can be written as

$$\lambda A_0 \frac{\pi\alpha}{\sin(\pi\alpha)} x + \lambda \sum_{i=0}^{n} \sum_{k=0}^{i} c_i a_{ik}^{(\alpha)} x^{k+1-\alpha} = f(x). \quad (11)$$

Similarly by substituting (5) into (2) and by using (8) and (9) we get

$$A_0 \left(x^\alpha + \lambda \frac{\pi\alpha}{\sin(\pi\alpha)} x \right) + \sum_{i=0}^{n} c_i P_i(x) + \lambda \sum_{i=0}^{n} \sum_{k=0}^{i} c_i a_{ik}^{(\alpha)} x^{k+1-\alpha} = f(x). \quad (12)$$

To find the solution of the first kind Abel integral equation (1) or the second kind Abel integral equation (2) we collocate (11) or (12) at $(n+2)$ points, respectively. For suitable collocation points we use the shifted Legendre roots z_i $(i = 1,\dots,n+1)$ of $P_{n+1}(t)$ and additional point $z_0 = 1$. The resulting equation generates a set of $(N+2)$ linear algebraic equations which can be solved for the unknown coefficients c_j $(j = 0,\dots,n)$

and A_0. Consequently $y(x)$ given in (5) can be calculated.

4. Numerical Experiments

This section is devoted to computational results. We apply the method presented in this paper and solve several examples. Those examples are chosen whose exact solutions exist.

Test 1. Consider the first kind Abel integral equation [7, 13]

$$\int_0^x \frac{y(t)}{\sqrt{x-t}}\mathrm{d}t = \frac{2}{105}\sqrt{x}(105-56x^2+48x^3), \quad (13)$$

which has the exact solution $y(x) = x^3 - x^2 + 1$. For this problem we use (11) with $\alpha = 1/2$ and $n = 3$. We obtain

$$A_0 = 0, \quad c_0 = \frac{11}{12}, \quad c_1 = \frac{-1}{20}, \quad c_2 = \frac{1}{12}, \quad c_3 = \frac{1}{20}.$$

Therefore using (5) we have

$$\begin{aligned} y_3(x) &= \frac{11}{12}P_0(x) - \frac{1}{20}P_1(x) + \frac{1}{12}P_2(x) + \frac{1}{20}P_3(x) \\ &= \frac{11}{12}(1) - \frac{1}{20}(2x-1) + \frac{1}{12}(6x^2-6x+1) \\ &\quad + \frac{1}{20}(20x^3-30x^2+12x-1) \\ &= x^3 - x^2 + 1, \end{aligned}$$

which is the exact solution.

Test 2. In the second example, we solve the second kind Abel integral equation [7, 13]

$$y(x) = x^2 + \frac{16}{15}x^{5/2} - \int_0^x \frac{y(t)}{\sqrt{x-t}}\mathrm{d}t. \quad (14)$$

For this problem we use (12) with $\alpha = 1/2$ and $n = 2$. We obtain

$$A_0 = 0, \quad c_0 = \frac{1}{3}, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{6}.$$

Therefore using (5) we have

$$\begin{aligned} y_2(x) &= \frac{1}{3}P_0(x) + \frac{1}{2}P_1(x) + \frac{1}{6}P_2(x) \\ &= \frac{1}{3}(1) - \frac{1}{2}(2x-1) + \frac{1}{6}(6x^2-6x+1) \\ &= x^2, \end{aligned}$$

which is the exact solution.

Tab. 1. Computational results of the absolute error $|y(x) - y_n(x)|$ of Test 1.

x	$n=3$	$n=5$	$n=7$	$n=9$	$n=11$
0.0	$5.5\cdot10^{-2}$	$2.2\cdot10^{-2}$	$1.1\cdot10^{-2}$	$6.2\cdot10^{-3}$	$3.8\cdot10^{-3}$
0.1	$2.4\cdot10^{-3}$	$5.4\cdot10^{-3}$	$1.1\cdot10^{-5}$	$2.2\cdot10^{-5}$	$1.2\cdot10^{-7}$
0.2	$1.2\cdot10^{-3}$	$7.0\cdot10^{-5}$	$2.3\cdot10^{-6}$	$1.6\cdot10^{-5}$	$2.7\cdot10^{-6}$
0.3	$1.6\cdot10^{-4}$	$2.7\cdot10^{-5}$	$2.9\cdot10^{-5}$	$1.4\cdot10^{-6}$	$1.4\cdot10^{-6}$
0.4	$1.9\cdot10^{-4}$	$5.5\cdot10^{-5}$	$7.1\cdot10^{-6}$	$2.3\cdot10^{-6}$	$1.8\cdot10^{-6}$
0.5	$1.3\cdot10^{-4}$	$5.2\cdot10^{-5}$	$7.9\cdot10^{-7}$	$3.7\cdot10^{-6}$	$6.0\cdot10^{-7}$
0.6	$3.8\cdot10^{-5}$	$1.7\cdot10^{-5}$	$6.6\cdot10^{-6}$	$1.3\cdot10^{-6}$	$1.0\cdot10^{-6}$
0.7	$1.4\cdot10^{-4}$	$2.4\cdot10^{-5}$	$6.2\cdot10^{-6}$	$1.5\cdot10^{-6}$	$6.0\cdot10^{-7}$
0.8	$1.3\cdot10^{-4}$	$1.1\cdot10^{-5}$	$2.6\cdot10^{-6}$	$1.6\cdot10^{-6}$	$6.3\cdot10^{-7}$
0.9	$6.3\cdot10^{-5}$	$1.5\cdot10^{-5}$	$3.4\cdot10^{-6}$	$1.0\cdot10^{-6}$	$4.8\cdot10^{-7}$
1.0	$4.8\cdot10^{-5}$	$9.7\cdot10^{-6}$	$2.8\cdot10^{-6}$	$1.0\cdot10^{-6}$	$4.5\cdot10^{-7}$

Test 3. In this example we apply the new method to find the solution of the singular Volterra integral equation [7]

$$y(x) = 2\sqrt{x} - \int_0^x \frac{y(t)}{\sqrt{x-t}}\mathrm{d}t. \quad (15)$$

The exact solution of this problem is $y(x) = 1 - \mathrm{e}^{\pi x}\mathrm{erfc}(\sqrt{\pi x})$, where $\mathrm{erfc}(x)$ is the complementary error function defined by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{x}}\int_x^\infty \mathrm{e}^{-u^2}\mathrm{d}u.$$

In Table 1 we present the absolute error $|y(x) - y_n(x)|$ for some values of x using the present method with $n = 3,5,7,9,11$. From Table 1 we see that the approximate solution computed by the present method converges to the exact solution. In Fig. 1, the absolute error function $|y(x) - y_n(x)|$ is plotted for $n = 7$ and $n = 10$.

Test 4. Consider the linear Volterra integral equation with algebraic singularity, presented in [13 – 15],

$$y(x) = \frac{1}{2}\pi x + \sqrt{x} - \int_0^x \frac{y(t)}{\sqrt{x-t}}\mathrm{d}t, \quad (16)$$

with the exact solution $y(x) = \sqrt{x}$. For this example we use (12) with $\alpha = 1/2$ and $n = 1$. We have $A_0 = 1$, $c_0 = 0$, $c_1 = 0$. Therefore using (5) we get the exact solution of this example.

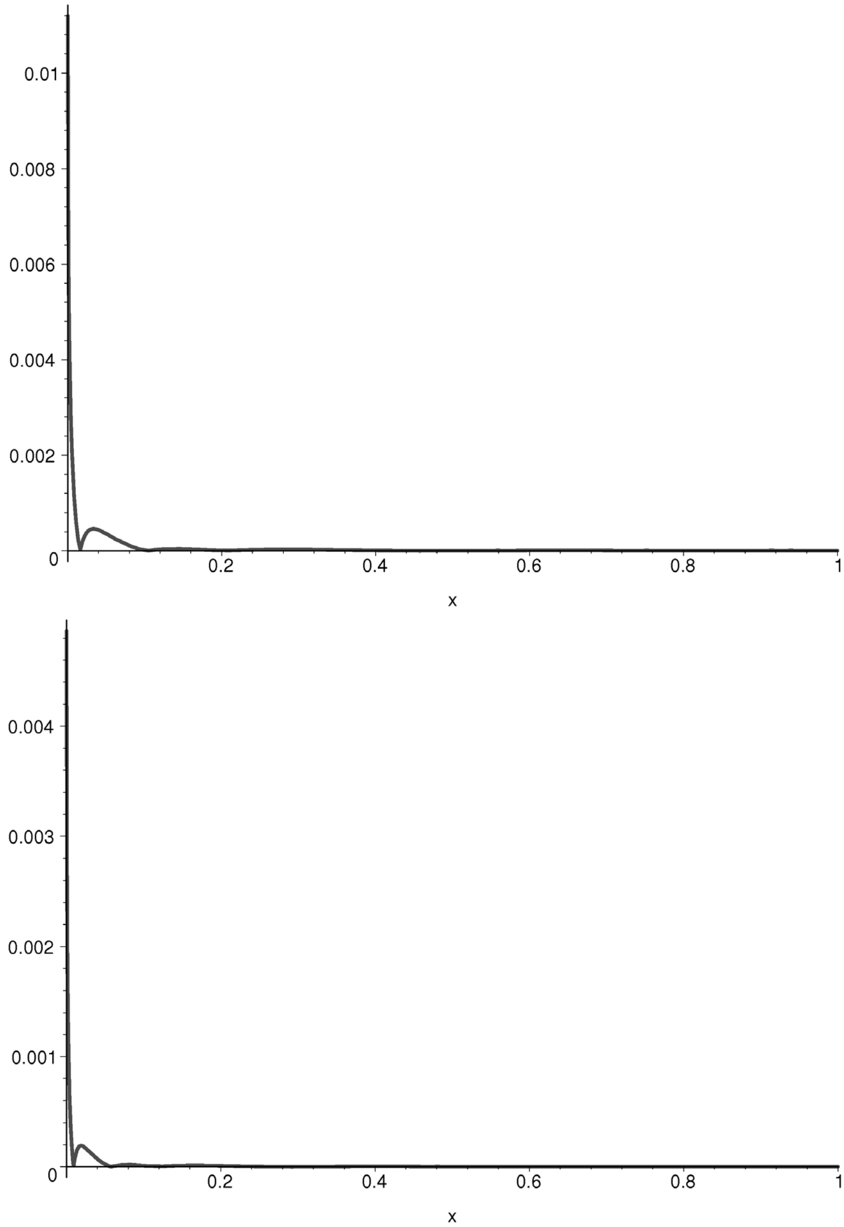

Fig. 1. Plot of the absolute error of Test 3 for $n = 7$ (top) and $n = 10$ (bottom).

Test 5. In this example we consider the second kind Abel integral equation with $\alpha = 1/3$, presented in [2],

$$y(x) - \frac{1}{10}\int_0^x \frac{y(t)}{(x-t)^{1/3}}\mathrm{d}t = x^2(1-x)^2 - \frac{729}{15400}x^{14/3} + \frac{243}{2200}x^{11/3} - \frac{27}{400}x^{8/3}. \tag{17}$$

For this example using (12) with $\alpha = 1/3$ and $n = 4$, we obtain $y(x) = x^4 - 2x^3 + x^2$, which is the exact solution of the problem.

Test 6. Consider the Volterra integral equation [16, 17]

$$y(x) = \frac{1}{\sqrt{x+1}} + \frac{\pi}{8} - \frac{1}{4}\sin^{-1}\frac{1-x}{1+x} - \frac{1}{4}\int_0^x \frac{y(t)}{\sqrt{x-t}}\mathrm{d}t \tag{18}$$

Table 2. The error $\|y-y_n\|_s$ for $s=1,2$ and some values of n of Test 6.

n	$\|y-y_n\|_1$	$\|y-y_n\|_2$
2	$1.02\cdot10^{-3}$	$1.15\cdot10^{-3}$
4	$2.72\cdot10^{-5}$	$3.10\cdot10^{-5}$
6	$7.24\cdot10^{-7}$	$8.26\cdot10^{-7}$
9	$3.20\cdot10^{-9}$	$3.68\cdot10^{-9}$
12	$1.45\cdot10^{-11}$	$1.41\cdot10^{-11}$
15	$6.71\cdot10^{-14}$	$7.78\cdot10^{-14}$

with the exact solution $y(x)=1/\sqrt{x+1}$. In Table 2 the error $\|y-y_n\|_s$ is illustrated for $s=1,2$ and $0\le x\le 1$ and some values of n. In this example we achieved a very good approximation with the exact solution of the equation by using only a few terms of shifted Legendre polynomials.

The numerical results obtained in this section demonstrate that the present method is capable of solving Abel's integral equations (1) and (2) and can be considered as an efficient method. Note that we have computed the numerical results by Maple programming.

Interesting applications of some integral equations are given in [18–20].

5. Conclusion

We presented a numerical scheme for solving Abel's integral equations of the first and second kinds. Our method consists of reducing Abel's integral equations to a set of linear algebraic equations by expanding the approximate solution as shifted Legendre polynomials with unknown coefficients. The obtained results showed that this approach can solve the problem effectively, and it needs less CPU time. The new described technique produces very accurate results even when employing a small number of collocation points.

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